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What We Can Learn About Irreversibility from a Cosmological Toy Model

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The main problems of the theory of irreversible processes are reviewed. It is conjectured that their solutions may be found in cosmology. A simple cosmological model showing interesting features that may lead to the answer to these problems is computed.

1. INTRODUCTION

In this paper we will not demonstrate anything, but we will show many interesting facts. In effect, we will only use approximate solutions and numerical calculations, so nothing that we say below is exact and rigorous. It is just a set of heuristic reasonings that must be considered as only a provisional approach to the problem. Precisely, we will consider and answer many of the typical questions formulated in the theory of irreversible process using as scenario a classical cosmological toy model, a Robertson–Walker closed geometry coupled with a homogeneous scalar field by an Einstein equation. Our aim is to show that all these questions can be only answered in a *global* system: the universe. All answers related to a *local* system, like that within a laboratory, are necessarily incomplete and raise many objections. This important fact can be seen using even the most elementary model of the universe like the one of this paper.

In the following subsections we list the main historical questions related to this subject.

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1.1. Zermello's Objection [1]

For a bounded system (as those in the laboratory) the Poincaré recurrence theorem [2] proves that the system will come back as close as we want to its initial position in phase space if we wait long enough. Of course this fact prevents any irreversibility. Even if in bounded systems with many degrees of freedom the recurrence time is extremely huge [3], we still have a theoretical objection. Also, if the system is open (an open universe), the objection disappears. But even if the system is closed, like the closed universe of our model, if it expands forever the recurrence time would be infinite, allowing the presence of irreversibility. This would be the case of our model for adequate initial conditions.

1.2. Loschmidt's Objection [4]

Any irreversible process begins in an unstable state. Loschmidt asks for the reasons of these initial unstable states. It is clear that within a laboratory and in a finite period of time it is impossible to find these causes. On the contrary, if we take into account the whole evolution of the universe, considered as a *global Reichenbach system* [5], the only instability that we must explain is the initial unstable low-entropy state of the universe. Our model begins in a unstable state for adequate initial conditions.

This is the real cause of irreversibility according to R. Feynman, who wrote, "For some reason, the universe at one time had a very low entropy for its energy content, and since then the entropy has increased. So that is the way towards the future. This is the origin of *all irreversibility*" [6].

1.3. The Ergodic Hypothesis

Thermodynamics is essentially based on the ergodic hypothesis: for each energy there is only one equilibrium state. Precisely,

$$\rho_* = \rho_*(H, \mathbf{P}, \mathbf{L}) \tag{1.1}$$

where ρ_* is the equilibrium classical density, H the energy, $\mathbf{P}$ the global linear momentum, and $\mathbf{L}$ the global angular momentum. But, as these two last quantities can be eliminated by changing the reference system [7], the last equation really reads

$$\rho_* = \rho_*(H) \tag{1.2}$$

This would be the case if H [and its functions like (1.2)] were the only constants of the motion, but sometimes there are other independent constants of the motion: H_1, H_2, . . . (namely the radii of the tori of the phase space). Nevertheless we will see that in our cosmological model only H is a constant of the motion (because all tori are broken by the interaction), so Eq. (1.2) and the ergodic hypothesis are valid.

For a fixed $H = const$, Eq. (1.2) leads to the microcanonical ensemble for our closed system. If the system is decomposed into many open systems, for each one of these systems, since the rest of the systems act as a thermostat, the canonical ensemble can be found, allowing the definition of temperature and the introduction of thermodynamics [8].

1.4. Dynamics vs. Thermodynamics

A very important question is the problem of the unification of the time-symmetric dynamical laws with the time-asymmetric thermodynamic laws. In fact, it is reasonable to think that thermodynamic laws could be demonstrated using the classical or quantum dynamical laws. But, it seems that this is not the case for the second law of thermodynamics, that says that entropy increases, in irreversible evolutions, leading the system to a state of thermodynamical equilibrium or maximal entropy. This problem can be stated as follows:

(i) Liouville equation is the time-symmetric evolution equation for classical distribution functions or quantum density matrices ρ.

(ii) This equation prevents the definition of any function of ρ: F(ρ) (constructed only with ρ and mathematical elements of the Liouville-phase space) such that $F(\rho) > 0$, namely it is impossible, as a consequence of the Liouville theorem, to define a Lyapunov variable, i.e., a growing function of ρ; e.g., the volume or the support of a characteristic distribution function ρ is time constant, Gibbs and conditional entropies are time constant [2], etc.

(iii) Nevertheless we actually see that the evolution leads the system to a thermodynamic equilibrium with a maximal entropy stationary state ρ_*.

Therefore the problem is to combine the Liouville theorem with the obvious fact that usual physical systems have a tendency to go to a thermodynamic equilibrium. The solution of the problem is based on a theorem by Mackey and Lasota [2]:

Theorem. Let $S(t)$ be an ergodic transformation, with stationary equilibrium density ρ_* [of the associated Frobenius–Perron operator $P(t)$ in a phase space of finite ρ_*-measure[4]]. Then if $S(t)$ is ρ_*-mixing if and only if $P(t)\rho$ is weakly convergent to ρ_*, i.e.,

[4] This theorem will be used on the matter content of the universe, in our case, the homogeneous matter field. In principle, the corresponding phase space has not finite measure. We will come back to this problem in a moment.

$$\lim_{t\to\infty}(P(t)\rho|g) = (\rho_*|g) \tag{1.3}$$

for all bounded measurable functions g.

That is, if the time evolution in phase space is $S(t)$ and the corresponding time evolution of the distribution functions is $\rho(t) = P(t)\rho(0)$, and this evolution is mixing, a chaotic property of evolutions that we will define below, and if there is an equilibrium density such that $P(t)\rho_* = \rho_*$, then Eq. (1, 2) can be proved.

But

$$\lim_{t\to\infty} P(t)\rho \neq \rho_* \tag{1.4}$$

In fact, as we will see, in many cases this limit does not even exist. Therefore we have a weak limit, but we do not have a strong limit (i.e., a limit in the norm).

Nevertheless we never see or measure ρ. What we see and measure are mean values of physical quantities O such that

$$\langle O\rangle_\rho = (\rho|O) = \int \rho(q, p)O(q, p)\, dq\, dp \tag{1.5}$$

Thus what we actually see is that

$$\lim_{t\to\infty}\langle O\rangle_\rho = \langle O\rangle_{\rho_*} \tag{1.6}$$

In fact, all the mean values of the physical quantities go to their equilibrium mean values if the evolution of the system is ρ_*-mixing. So the solution of the problem is quite easy:

(i) The Liouville theorem is embodied in Eq. (1.4): the system does not go (strongly) toward the equilibrium states.

(ii) The tendency toward equilibrium is embodied in Eq. (1.6): the mean values of all the physical quantities go to their equilibrium values.

Clearly these facts are not contradictory. Once this tendency is found, it is not difficult to define a growing entropy (using, e.g., coarse-graining or other methods). As chaotic-mixing systems are very frequent in the universe, the problem is essentially solved, but it would be more soundly based if we could prove that the whole universe (or better, its matter content) is a mixing system. A system defined in a finite measure phase space is mixing if it is ergodic and is endowed with a continuous frequency spectrum [9]. We will show that our model is ergodic (since the only constant of the motion is H) and that it has a continuous spectrum. Then it would be mixing if it was a system defined in a finite measure phase space, *but it is not*. In fact, the relevant thermodynamic system is the matter content of the universe (i.e., the matter field in our model) This subsystem is also ergodic (since the whole system itself is ergodic) and it has a continuous spectrum, nevertheless it is not defined in a finite measure phase space either. But we will show that it has a mixing behavior, since the support of any density expands forever (most probably toward a uniform distribution, since the only equilibrium distribution is a constant, but, of course, this is only a conjecture). Even if this is a delicate point, we think these facts could be used to solve the problem of this subsection.

Thus the main questions on irreversible processes would be answered in this way.

2. THE SYSTEM

We will study a toy cosmological system consisting of a closed homogeneous geometry of radius a coupled to a scalar matter field φ of mass m through a conformal coupling ($\xi = 1/6$), as already considered in refs. 10–13. The corresponding Einstein equation reads

$$H(a, p_a, \varphi, p_\varphi) = -\frac{1}{2}(p_a^2 + ka^2) + \frac{1}{2}(p_\varphi^2 + k\varphi^2) + \frac{m}{2}a^2\varphi^2 = 0 \tag{2.1}$$

where, in our case, $k = 1$ ($k = 0$ for a critical universe and $k = -1$ for an open universe). H can be considered as the Hamiltonian of a usual mechanical equation and Einstein equation $H = 0$ as a constraint that fixes the energy.

The corresponding motion equations are

$$\begin{aligned} a^{\bullet} &= -p_a, \qquad \varphi^{\bullet} = p_\varphi \\ p_a^{\bullet} &= a(1 - m\varphi^2) \\ p_\varphi^{\bullet} &= -\varphi(1 + ma^2) \end{aligned} \tag{2.2}$$

which yield the second order system

$$\begin{aligned} a^{\bullet\bullet} &= -a(1 - m\varphi^2) \\ \varphi^{\bullet\bullet} &= -\varphi(1 + ma^2) \end{aligned} \tag{2.3}$$

This system can be interpreted as that of two coupled oscillators with proper frequency equal to one and where the nonlinearity is introduced by the m term. We will consider a small mass $m = 10^{-3}$ (compared with the Planck mass). Even so, the terms $ma\varphi^2$ and $m\varphi a^2$ cannot be considered as perturbations because for large a and φ they cannot be neglected.

We list the main properties of the system below.

2.1. Scaling Law

The change of variables

$$a' = \lambda a, \qquad \varphi' = \lambda \varphi, \qquad m' = \frac{m}{\lambda^2} \tag{2.4}$$

leaves the system invariant. This scaling law can be used to consider systems with different mass *m*.

2.2. Fixed Points and Stability

The only fixed point of the system is ($a = Pa = \phi = P\phi = 0$). Then:

(i) When $m = 0$ the system becomes linear; precisely, that of two uncoupled oscillators that oscillate with frequency $\omega = 1$. In this case the fixed point is stable—precisely, a center.

(ii) When $m \neq 0$, the system loses its linearity. Then we can analyze the stability of the fixed point by the method of first approximation [14, 18, 20], which is valid for a neighborhood of the origin. Then it can be shown that, for an initial condition in this neighborhood, the fixed point can be considered a center. Nevertheless for other initial conditions the nonlinear terms become important and this stability analysis is no longer valid. Thus we can see how the interactions play an important role in the properties of the solutions [14, 15].

2.3. Weak Interaction Solutions

As the system depends on the mass *m* in a continuous way, it is possible to find, for a small *m*, periodic solutions that can be expanded in power series of *m* [14, p. 156]—precisely, when $m \ll 1/\varphi^2$ and $m \ll 1/a^2$. Thus, the condition $m \ll 1$ is not a sufficient one to allow this kind of expansion (see below).

2.4. Approximate Solutions

Due to the nonlinear terms it is, of course, impossible to find closed analytical solutions for the system (2.3). But since the system is continuous in *m*, *a*, and φ it is possible to find an approximate solution for small time intervals. Then we can study these solutions and find their properties, which will drastically change for different initial conditions. After the approximation [17], the system reads

$$
\begin{aligned}
a^{\bullet\bullet} &= -a_j(1 - m\hat{\varphi}_j^2) \\
\varphi^{\bullet\bullet} &= -\varphi_j(1 + m\hat{a}_j^2)
\end{aligned}
\tag{2.5}
$$

for a time t such that $t_j - \varepsilon < t < t_j + \varepsilon$, where $|\varepsilon| \ll 1$, t_j is an arbitrary time, and $\hat{a}_j$ and $\hat{\varphi}_j$ are the following constants:

$$
\hat{a}_j = \frac{1}{2}[a(t_j + \varepsilon) + a(t_j - \varepsilon)]
$$

$$
\hat{\varphi}_j = \frac{1}{2}[\varphi(t_j + \varepsilon) + \varphi(t_j - \varepsilon)] \tag{2.6}
$$

Then, the approximate solution of Eq. (2.5_2) is always periodic, and the harmonics have the frequencies

$$
\omega_j = \sqrt{1 + m\hat{a}_j^2} \tag{2.7}
$$

while the solution of Eq. (2.5_1) is

(i) periodic for $\hat{\varphi}_j^2 < 1/m$, (ii) polynomial if $\hat{\varphi}_j^2 = 1/m$, and (iii) exponential if $\hat{\varphi}_j^2 > 1/m$.

This simple analysis shows the appearance, for some initial conditions, of an exponential behavior if the interaction term becomes big enough. This fact is shown by numerical computations and is essential for the stated program in the introduction. In fact, it solves Zermello's objection (Section 1.1).

We can also transform Eq. (2.3_1) into an approximate equation [16]; then we have

$$
\begin{aligned}
a^{\bullet\bullet} &= -a_j(1 - m\hat{\varphi}_j^2) \\
\varphi^{\bullet\bullet} &= -\varphi_j(1 + ma^2)
\end{aligned}
\tag{2.8}
$$

Then for $\hat{\varphi}_j^2 > 1/m$ we have

$$
\begin{aligned}
\varphi_j(t) = (&-e^{-k_j t} k_j \cos t + 4k_j \cos t\, c_{1_j} + c_{2_j} k_j^3 \cos t \\
&- 2e^{k_j t} \sin t - c_1 4k_j \sin t - k_j^2 c_{1_j} \sin t)/(4k_j + k_j^3)
\end{aligned}
\tag{2.9}
$$

where $k_j^2 = (1 - m\hat{\varphi}_j^2)$.

for $\hat{\varphi}_j^2 < 1/m$ we have

$$
\begin{aligned}
\varphi(t) = c_{2_j} \cos t &+ \cos t \cos 2k_j t/8(1 - k_j^2) - t \sin t/(4 \\
&- c_{1_j} \sin t) + \sin t \sin 2k_j t/8k_j(1 - k_j^2)
\end{aligned}
\tag{2.10}
$$

These equations show that the described behavior also appears for higher approximation orders.

3. NUMERICAL RESULTS

The system was solved using the Runge-Kutta method up to the fourth order [17, 19, 21]. The integration steps were fixed in such a way that $|H| < 10^{-8}$. Recall that we have the constraint $H = 0$. Taking into account this constraint, we will choose the initial condition in phase space as

$$p_a(0) = \varphi(0) = p_\varphi(0) = \theta \tag{3.1}$$

Increasing θ, the energy of each oscillator will also increase.

To find the most interesting solutions we have projected them over the plane (a, p_a) [20] and we have computed the power (frequency f) spectrum [15] in order to characterize the behavior of the system. We have the following results:

(a) The case $m = 0$. If there is no interaction, the analytical solutions have a constant frequency corresponding to $\omega = 1$. [The numerical result of the power spectrum is strongly peaked at $f = 1/(2\pi)$.] Moreover, there are two independent constants of the motion:

$$H_1 = \frac{1}{2}(p_a^2 + a^2) = const, \qquad H_2 = \frac{1}{2}(p_\varphi^2 + \varphi^2) = const \tag{3.2}$$

These are the radii of the tori of the unperturbed model.

(b) The case $m \neq 0$—precisely, $m = 10^{-3}$.

(b1) When $\theta = 10$, the projection of the solutions over the plane (a, p_a) seems to be an ellipse, namely a regular figure with a constant shape (Fig. 1). Then the corresponding tori are not broken and the two equations (3.2) are still approximately valid. $a(t)$ looks like a modulated oscillation,

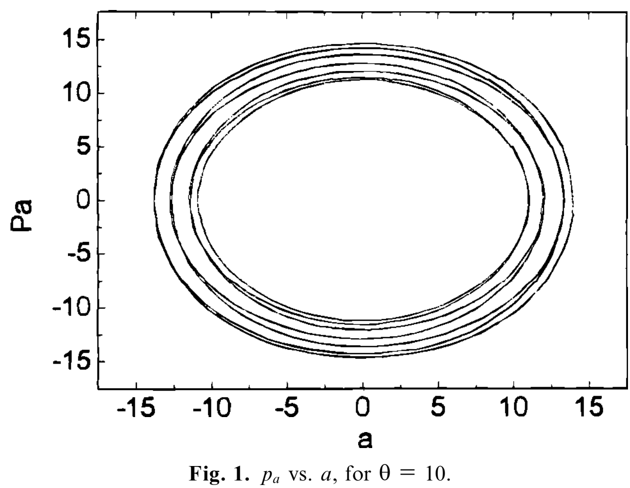

Fig. 1. p_a vs. a, for $\theta = 10$.

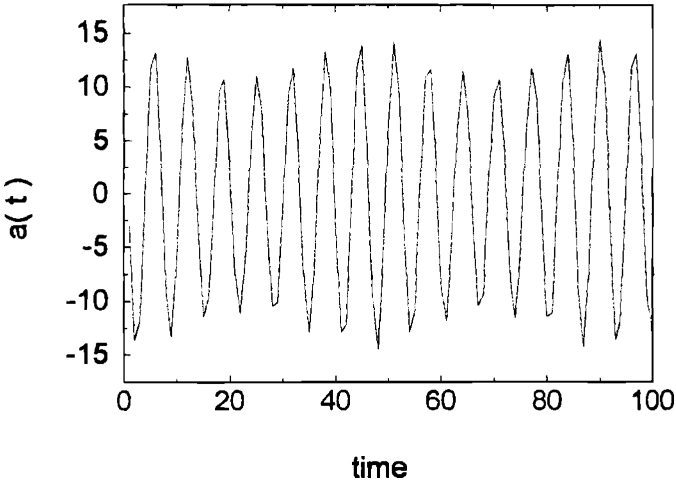

Fig. 2. The $a(t)$ function, for $\theta = 10$.

and the power spectrum looks like a discrete one, but new frequencies appear (Figs. 2, 3) In Fig. 4 we can see that the interaction term can be considered as a perturbation on the Hamiltonians of each oscillator, which may be considered as weakly coupled. The behavior of H_1 is similar to that of H_2, but these two functions are no longer constant, and they show an exchange of energy with a modulated oscillation.

(b2) When $\theta = 20$, the energy of each oscillator is increased and the interaction ceases to be just a perturbation. The power spectrum still looks

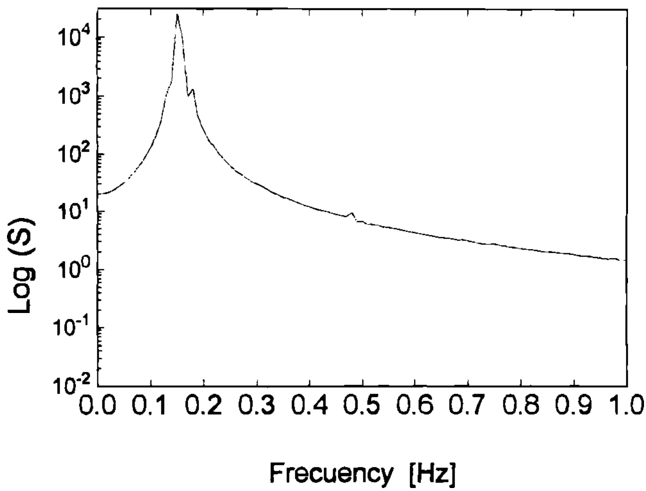

Fig. 3. The power spectrum $S(t)$, for $\theta = 10$.

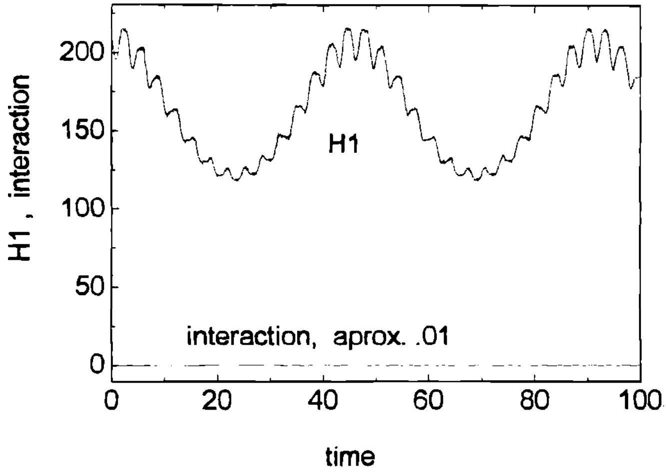

Fig. 4. H_1 vs. t, for $\theta = 10$.

discrete, but more frequencies appear (Fig. 5) The projection of the solution over the plane (a, p_a) still has an elliptic shape (Fig. 6), but now a more diffused one, showing that the tori are still not badly broken. The function $a(t)$ ceases to have a modulated oscillation shape, and new harmonics appear (Fig. 7). Figure 8 clearly shows the importance of the interaction term and the loss of periodicity of H_1 and H_2.

(b3) When $\theta = 30$, using numerical methods, we find that there is a critical θ ($\theta_{cr} = 23, 25$). This critical θ can be considered as a critical mass if we use Eq. (2.4). Above this threshold the system radically changes its behavior. In the case $\theta > \theta_{cr}$ we find the following:

(i) The appearance of a continuous frequency spectrum (Fig. 9).

(ii) The projections on the plane (a, p_a) are completely erratic, thus the tori are completely broken (Fig. 10), the only constant of the motion is H, and the ergodic hypothesis is valid (see Section 1.3 and Fig. 11). These results show that we are in a mixing regime, therefore dynamics can be unified with thermodynamics.

(iii) $a(t)$ shows an exponential behavior [Fig. 12, last part of the curve and also Eq. (2.9)]. The universe is expanding and there is no possible recurrence. Zermello's objection has been overcome.

(iv) The system is unstable, so the primeval instability has appeared, defeating, via Reichenbach's global diagram, Loschmidt's objection.

(v) Strong resonances appear, preventing the use of the perturbative method to find the approximate solution (Fig. 11), but all these results are consistent with the local Euler approximation introduced in Eq. (2.8).

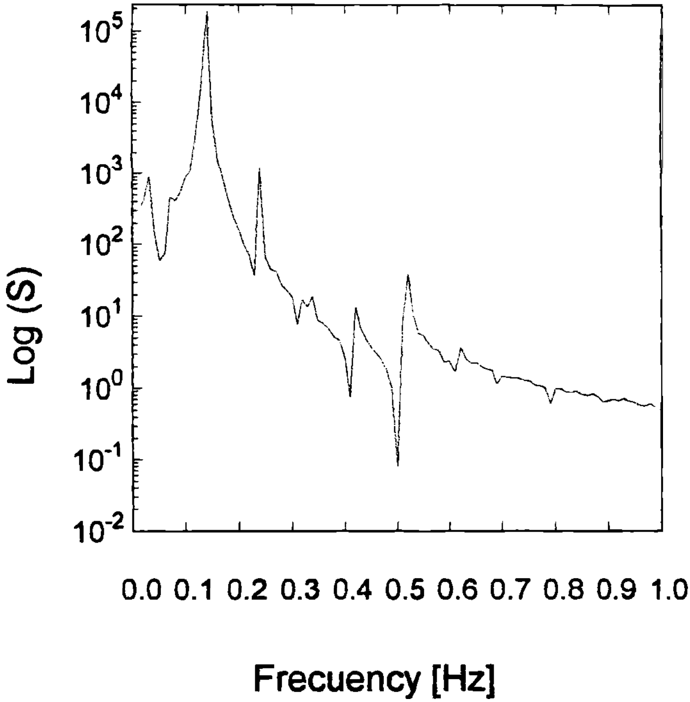

Fig. 5. The power spectrum $S(t)$, for $\theta = 20$.

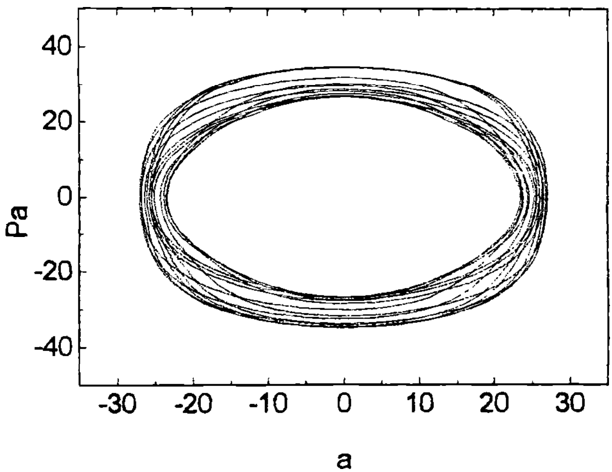

Fig. 6. p_a vs. a, for $\theta = 20$.

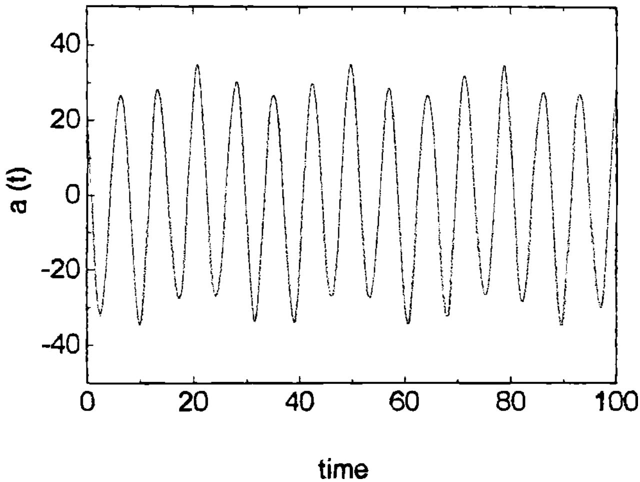

Fig. 7. The $a(t)$ function, for $\theta = 20$.

(vi) In Figs. 13 and 14 we see how a set of three solutions at time $t = 0$ (solid, pointed, and dotted lines) become far apart at later times, showing that the support of any density expands with time. This fact sustains the conjecture of Section 1.4.

The existence of chaos in the model is also exactly proved in refs. 11 and 13. Beautiful Poincaré diagrams showing the broken tori can be found in ref. 12. So all the results agree.

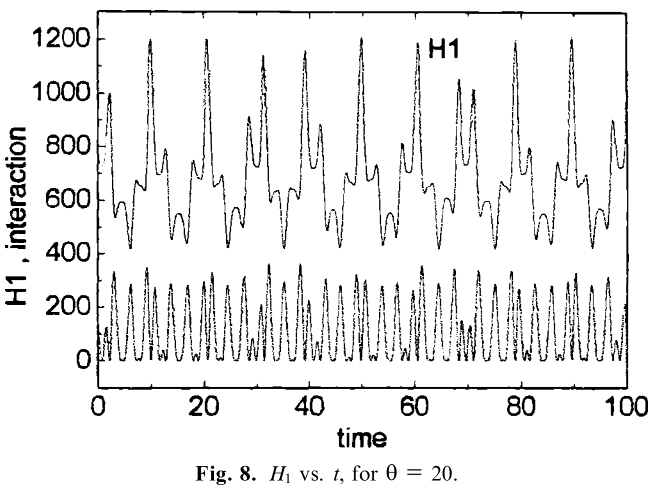

Fig. 8. H_1 vs. t, for $\theta = 20$.

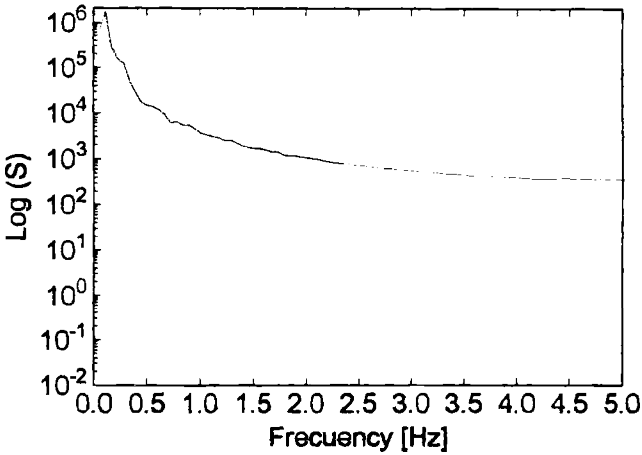

Fig. 9. The power spectrum $S(t)$, for $\theta = 30$.

4. CONCLUSION

We have presented only a sketch of a panorama that must be worked out to arrive at the definitive solution of the problems stated in the Introduction. Nevertheless, we believe that the main lines of this project are already established in this paper. Moreover, we have proved the importance of a global treatment to find conclusive solutions.

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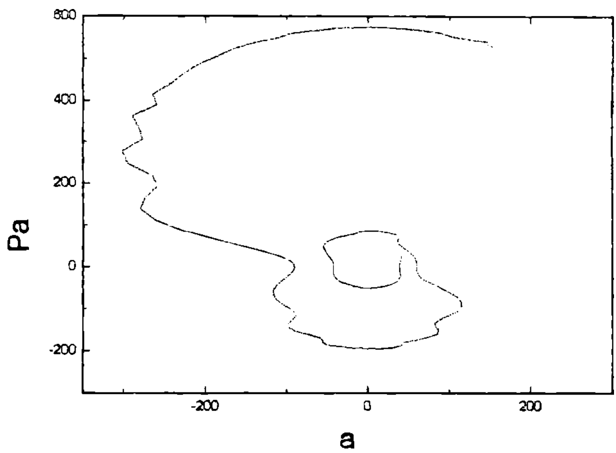

Fig. 10. p_a vs. a, for $\theta = 30$.

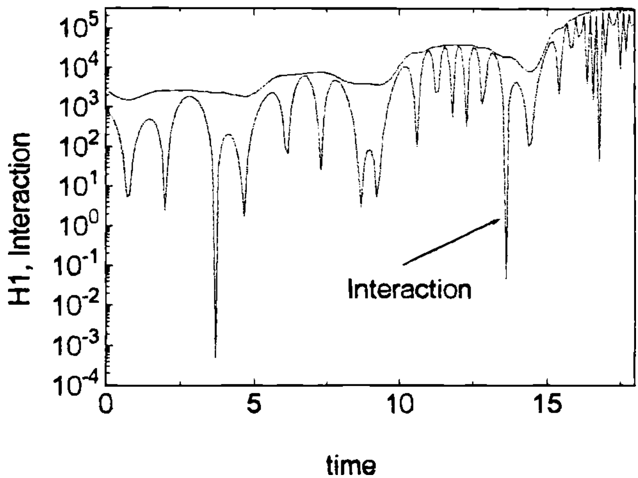

Fig. 11. H_1 vs. t, for $\theta = 30$.

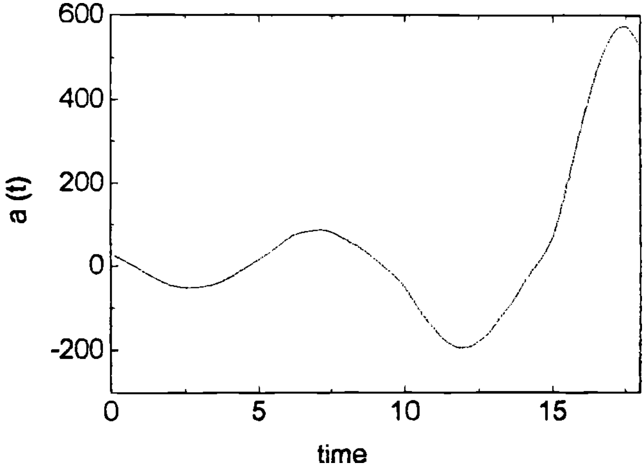

Fig. 12. The $a(t)$ function, for $\theta = 30$.

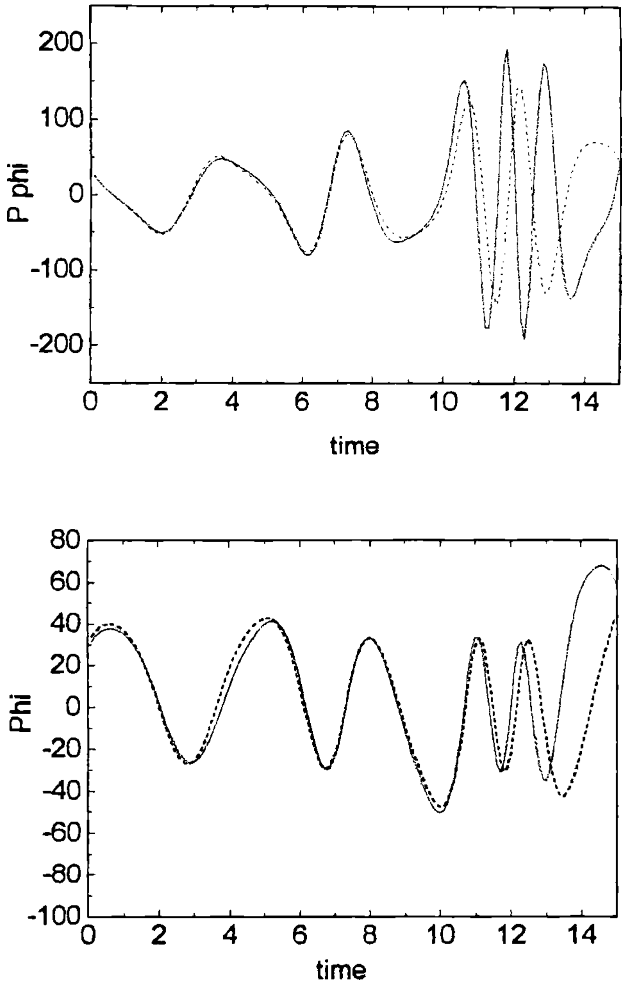

Fig. 14. φ vs. *t* for different initial conditions.

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